Multivortices in Abelian Higgs Models

Numerical and Analytical Results

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References

- The sections "Classical Field Theory" and "Solitons - Domain Walls and Vortices" are mainly based on [7], [5] and a little on [9], while [10] states the properties of the electric field tensor. The [6] has only been used for understanding the concepts.
- The sections "Domain Walls and Multi-vortices" and "The Differential Equations" is based on [1] solely.
- The section "The Numerical Method" is based mostly on [8] and the results of [1].
- In the last section of "Linking to reality - Superconductivity" I have used a few results from [7].

Abstract in Danish

Et *felt* er i bund og grund en funktion der afhænger kontinuert af sted og tidskoordinater. Mange størrelser kan beskrives ved felter: temperatur, tæthed, densitet etc. Et felt beskrives ved en *Lagrangetæthed* mens bevægelsesligningerne fremgår af Euler-Lagrange ligningerne.
Generelt kan en Lagrange tæthed være invariante under forskellige transformationer af felterne. Man kan udnytte dette ved at indføre *gauge felter*. Er der tale om et felt ϕ med en Lagrangetæthed invariant under $U(1)$-transformationer vil det tilhørende gauge felt blot være det gammelkendte fotonfelt $A_\mu = (V/c, A_x, A_y, A_z)$. I dette tilfælde er Lagrange tætheden

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\partial_\mu + ieA_\mu)\phi^*(\partial^\mu - ieA^\mu)\phi + V(|\phi|)$$

Lad nu ϕ udvise en form for symmetri. I vakuum vil feltet ϕ nærme sig et af nulpunkterne for potentialet V. Hvis potentalet har flere nulpunkter kan vi specificere ikke-trivielle grænsebetingelser i uendelig $(x, y, z) \to \infty$. Dette kan resultere i et spontant symmetribrud, hvor vakuum-energien ikke længere er nul, og feltet $\phi \neq 0$.
Man kalder feltkonfigurationerne for *solitoner*. Vi skal beskæftige os med to slags solitoner. *DDomæne vægge* kan kvalitativt skrives som en uendelig udstrakt grænseflade af endelig tykkelse mellem to faser, i vores tilfælde den rene elektromagnetiske fase hvor $\phi = 0$ og Higgs fasen $\phi = \phi_0$. En *Abrikosov-Nielsen-Olesen vortexlinje* kan kvalitativt forstås som en cylinder indeholdende en kvantiseret magnetisk fluks.
I artiklen *Multi-Vortices are Wall vortices - A numerical Proof* betragter Bolognesi og Gudnason en domæne væg som er foldet sammen til en cylinder med en kvantiseret magnetisk fluks Φ indeni. Ideen er a vise at i grænsen $\Phi \to \infty$ vil en sådan væg-vortex være identisk til en ANO vortexlinje, og derfor vil energitætheden per overfladeareal (også kendt som spændingen) være identisk, så

$$\lim_{n\to\infty} T_V(n) = T_{MIT}(n) = 2\sqrt{2}\pi\frac{\sqrt{\epsilon_0}}{e}n$$

Bevægelseslingingerne for en ANO vortexlinje, hvor f er det reskalerede felt, er som følger

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - n^2\frac{(1-A)^2}{r^2}f - \frac{1}{2}\frac{\delta V}{\delta f} = 0$$
$$\frac{d^2A}{dr^2} - \frac{1}{2}\frac{dA}{dr} + 2e^2(1-A)f^2 = 0$$

Disse kan desværre ikke løses analytisk. Numerisk kan vi udvælge et interval og opdele det i N punkter. I hvert punkt kan vi udregne værdierne for de tilsvarende *finite difference* ligninger. Hvis disse var lineære var det nu blot et spørgsmål om at løse en simpel matrixligning, men i dette tilfælde må vi benytte en mere besværlig metode hvor vi langsomt forsøger at iterere os frem til en løsning.
Vi betragter kun betragter et ϕ^4 potential. De numeriske resultater er stort set identiske med Bolognesis og Gudnasons resultater, og efterviser netop den påståede sammenhæng mellem væg-vortexlinjer og vortexlinjer.
Udover de resultater Bolognesi og Gudnason betragter, har jeg valgt også at betragte to ekstremtilfælde, både for at sikre at den numeriske proces er stabil og for at undersøge om disse resultater også stemmer overens med de andre resultater. Konklusionen er at den numeriske proces er udmærket stabil, dog halter lidt, samt at resultaterne er i fuld overensstemmelse med det forventede.

1 The structure of this report

In this project I basically just go through and explain the paper *Multivortices are Wall vortices: A numerical proof* written by Stefano Bolognesi and Sven Bjarke Gudnason.
The first par of the project is a description of the theoretical subjects forming the basis for the paper. I introduce the Lagrangian Formalism and the theory of fields. Since we will only consider an $U(1)$ Higgs model we will not go deeper into the concept of gauge field theory than necessary. Then the concept of spontaneous symmetry breaking is explained and it is shown that it will lead to the formation of solitons. Last the domain wall soliton and the Abrikosov-Nielsen-Olesen vortex solutions are described in mathematical detail along with the BPS case.
The second part of the project is more closely based on the paper. Here I describe the conjecture that a wrapped up wall vortex is similar to a vortex solution in the large magnetic flux limit. Following the lines of the paper the necessary mathematics such as rescaling and boundary conditions are calculated. The core of the project is then the numerical analysis, done in Fortran 90 (whereas Bolognesi and Gudnason used Maple). The numerical method developed for solving this problem is then described along with the actual code. The results obtained with the method are then discussed luckily being very close to the results of the paper. I have supplied the results with the cases of two very extreme examples. At last I try linking the results obtained to superconductivity.

2 Notation

In field theory tensor notation is used extensively, and it is therefore fitting to introduce the conventions here. In the following we will use the metric tensor

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

with greek indicies running over $0, 1, 2, 3$ (or t, x, y, z), and roman indices $i, j, \ldots$ denoting only spatial dimensions (x, y, z). We will be using Einstein notation, summing over repeated indices. Therefore we can write the four-vectors as

$$x^\mu = (x^0, \mathbf{x})x_\mu = g_{\mu\nu}x^\nu = (x^0, -x)$$

And last we will use the derivative operator defines as

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla\right)$$

3 Classical Field Theory

3.1 Introduction

Simple highschool mechanics normally deals with particles having a mass m, a position x and a velocity v. Using these quantities we can calculate the energy and momentum of the system to predict

it's behaviour. But quantities such as density, air pressure, temperature, a quantum mechanical probability amplitude, electric field etc. cannot be described as discrete particles.
We are therefore led to the concept of a *field*. A field $\phi(t, \mathbf{x})$ describes entities depending continuously on the spatial dimensions and time, and like systems of discrete particles fields we can also attribute quantities like energy and tension to the field. Classical field theory describes the properties of fields in a classical setting i.e. non semi-quantum mechanical.

3.2 Lagrangian formalism

Let us first briefly recount the methods of Classical Mechanics. A system parametrised by the generalized coordinates g_i has kinetic energy T and potential energy V. From the Lagrangian of the system, $L(q_i, \dot{q}_i) = T - V$, we construct the action

$$S[q(\cdot)] = \int_{t_1}^{t_2} dt\, L(q, \dot{q})$$

Then we know that the action is stationary for the path taken by the system in q-space. So

$$\delta S = \int_{t_1}^{t_2} dt \delta L(q_i, \dot{q}_i) = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right] = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right] \delta q_i = 0$$

which yields the Euler-Lagrange equations of motion. Here we can also define the conjugate momentum p_i along with the Hamiltonian H

$$p_i(q, \dot{q}) \equiv \frac{\partial L}{\partial \dot{q}_i}$$
$$H(q, p) \equiv \dot{q}_i(q, p)p_i - L(q, \dot{q}_i(q, p)$$

The point of using the Lagrangian formalism instead of Hamiltonian formalism is that the Lagrangian is Lorentz invariant and we can thus include relativistic effects.
Consider now an abstract field $\phi(t, \mathbf{x})$. To keep things simple we will assume that the Lagrangian will depend only upon the field configurations ϕ and the derivative $\partial_\mu \phi$ so

$$L(\phi(t, \cdot), \dot{\phi}(t, \cdot)) = \int_V d^3x\, \mathcal{L}(\phi(t, \mathbf{x}), \partial_i \phi_r(t, \mathbf{x}), \dot{\phi}(t, \mathbf{x}))$$

Where $\mathcal{L}$ denotes the Lagrange density. In this case the action becomes

$$S[\phi(\cdot)] = \sum_\Sigma d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x))$$

And invoking the principle of least action we get

$$\delta S[\phi] = \int_\Sigma \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\partial_\mu \phi(x)\right] = \int_\Sigma \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right] \delta\phi(x)\, d^4x = 0$$

which also contains the Euler-Lagrange Equations for a field,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

Correspondingly we can define the conjugate momentum

$$\pi(t, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi)}$$

and a Hamiltonian density

$$\mathcal{H}(\phi, \partial_i \phi; \pi) = \dot{\phi}\pi - \mathcal{L}(\phi, \partial_\mu \phi)$$

More generally we can define a stress-energy tensor

$$T_{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \mathcal{L}\partial^\mu_\nu$$

A simple rewriting of the Hamiltonian density reveals that

$$\mathcal{H} = \dot{\phi}\pi_0 - \mathcal{L} = (\partial_0 \phi)\phi_0 + \eta_{00}\mathcal{L} = T_{00}$$

The rest of the terms in $T_{\mu\nu}$ shows different aspects of the system. The terms T_{ii} represents pressure or tension terms.

3.3 Gauge Symmetry

Consider a Lie group A of elements g represented by some complex matrices $A(g)$. Using these matrices we can transform some field

$$\phi \to A(g)\phi$$

We will denote A a symmetry group if any transformation $A(g)$ leaves the corresponding Lagrangian density invariant, thus exposing a symmetry of the system.
Taking the unitary group $U(N)$ consisting of unitary matrices U for which $U^\dagger U = 1$, the Lagrangian density for some system of fields $\phi = (\phi_1, \ldots, \phi_N)$

$$\mathcal{L}(\phi) = \sum_i^N \left[-\partial_\mu \phi_i^\dagger \partial^\mu \phi_i - V(\phi_i^\dagger \phi_)\right] \tag{3.1}$$

is obviously left invariant under such a unitary transformation. Thus $U(N)$ is a symmetry group of the system.
In case of $U(1)$ it is quite obvious that we can write the transformation like

$$\phi \to \mathrm{e}^{i\alpha(g)}\phi \tag{3.2}$$

And it is thus an abelian field theory[1].

3.4 Applying the U(1) Gauge Theory

The Lagrangian (3.1) left invariant by $U(1)$ can be put on a slightly different form

$$\mathcal{L}_0 = -(\partial_\mu \phi)^*(\partial_\mu \phi) - V(\phi^* \phi)$$

[1]When invoking two $U(1)$ field transformations the result will not depend on the order of which the transformation were applied. That the transformations commutate is the same as saying that they are abelian.

Forcing invariance under local gauge transformations we introduce a gauge field $A_\mu(x)$ which transforms together with ϕ,

$$\phi(x) \to \mathrm{e}^{i\alpha(x)}\phi \tag{3.3}$$
$$A_\mu(x) \to A_\mu + \partial_\mu\alpha(x) \tag{3.4}$$

Note that A is a vector field having components in space-time. It is also useful to define a covariant derivative

$$D_\mu = \partial_\mu - ieA_\mu \tag{3.5}$$

Where e is some coupling constant - normally a charge. Transforming D_μ we get the simple relation

$$\begin{aligned} D_\mu\phi(x) &\to [\partial_\mu - iA_\mu - i\partial_\mu\alpha(x)]\,\mathrm{e}^{i\alpha(x)}\phi(x) &(3.6)\\ &= \mathrm{e}^{i\alpha(x)}D_\mu\phi(x) &(3.7)\end{aligned}$$

and we say that D_μ transforms covariantly. Thus the Lagrangian

$$\mathcal{L} = -(D_\mu\phi)^*(D_\mu\phi) - V(\phi^*\phi)$$

will be invariant under local gauge transformations.
We can identify the gauge field A_μ with the photon field (V, A_x, A_y, A_z), where V is a scalar potential and $\mathbf{A}$ is a vector potential. This comes from the fact that

$$\begin{aligned}\mathcal{L}(A_\mu) &= -\frac{1}{2}\left(\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu\right)\\ &= -\frac{1}{4}\left(\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A_\mu - \partial_\nu A_\mu\partial^\mu A^\nu + \partial_\nu A_\mu\partial^\nu A_\mu\right)\\ &= -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &(3.8)\end{aligned}$$

where we have defined the electromagnetic field tensor

$$F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Applying the Euler-lagrange equations we arrive at

$$\begin{aligned}&\partial_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = 0 \Leftrightarrow\\ &\partial_\nu F^{\mu\nu} = 0\end{aligned}$$

Which is actually identical to the homogeneous Maxwell equations when we identify V with the electrical potential and $\mathbf{A}$ with the magnetic vector potential. Rewriting $F^{\mu\nu}$ we obtain

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3\\ -E_1 & 0 & B_3 & -B_2\\ -E_2 & -B_3 & 0 & B_1\\ -E_3 & B_2 & -B_1 & 0\end{pmatrix}$$

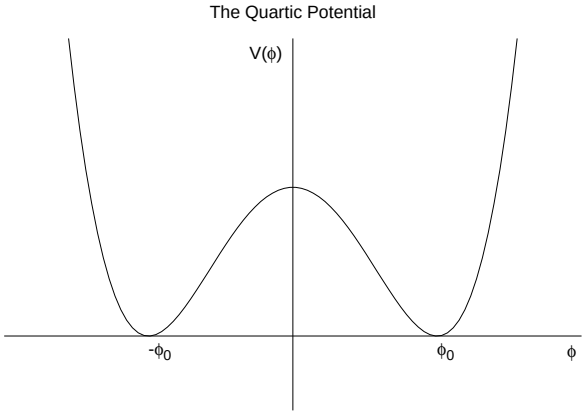

Figure 1: An example of a quartic potential $V(\phi) = (\phi^2 - \phi_0^2)^2$

3.5 Spontaneous symmetry breaking

Consider the Lagrangian of a U(1) phase in the case of a ϕ^4 potential $V(\phi^*\phi) = \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2$,

$$\mathcal{L} = -\partial_\mu\phi^*\partial^\mu\phi - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 \tag{3.9}$$

A vacuum state is a minimum of the potential $V(\phi^*\phi) = \mu^2\phi^*\phi + \lambda(\phi^*\phi)^2$. In case $\mu^2 > 0$ the vacuum is simply given by $\phi = 0$. On the other hand $\mu^2 > 0$ yields degenerate vacuum states

$$\phi = \mathrm{e}^{i\alpha}\phi_0, \text{ where} \qquad \phi_0^2 = -\frac{\mu^2}{2\lambda}$$

Since the Lagrangian (3.9) is invariant under global gauge symmetries we can choose a gauge such that the ϕ_0 is some real constant v. Expanding around this vacuum state,

$$\phi(x) = \phi_0 + \delta\phi(x) = \begin{pmatrix} v \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} s(x) \\ \pi(x) \end{pmatrix}$$

and rewriting $\mathcal{L}$ using the property $\mu^2 + 2\lambda v^2 = 0$ together with the expression for the displaced fields s and π,

$$\mathcal{L} = -\frac{1}{2}\left((\partial_\mu s)^2 + (\partial_\mu\pi)^2\right) - 2\lambda v^2 s^2 + \mathcal{L}_{int} \tag{3.10}$$

$$\mathcal{L}_{int} = -\sqrt{2}\lambda v s(s^2 + \pi^2) - \frac{\lambda}{4}(s^2 + \pi^2)^2 \tag{3.11}$$

In terms of quantum field theory, one refer to expressions like $1/2m^2\phi^2$ or $m^2\phi^2$ as mass terms. Thus we clearly see one massive excitation of mass $m_s^2 = 4\lambda v^2$ (the s field), and a corresponding massless excitation (the π field).
Let us instead consider an abelian gauge field $A_\mu(x)$. Then the Lagrangian density will look like

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\partial_\mu + ieA_\mu)\phi^*(\partial^\mu - ieA^\mu)\phi - \mu^2\phi^*\phi - \lambda(\phi^*\phi) \tag{3.12}$$

The case $\mu > 0$ will again lead to the vacuum case of $\phi = 0$. We see that in this case the Lagrangian will reduce to the electromagnetic case of (3.8), and therefore the case of $\phi_0 =$ is normally called the *Coulomb phase*.

If $\mu^2 < 0$ we will again have a spontaneous breaking of symmetry at $\phi = \phi_0$. This case is then referred to as the *Higgs phase*. Because of the local gauge symmetry for some $\alpha(x)$ the vacuum states will be characterised by

$$\phi_0(x) = \mathrm{e}^{i\alpha(x)}|\phi_0| \qquad A_{\mu 0}(x) = \partial_\mu \alpha(x) \tag{3.13}$$

Making a local gauge transformation rotating $\phi(x)$ to a real value and expanding the field around a vacuum state,

$$\phi(x) \to \mathrm{e}^{i\alpha(x)}\phi(x) = |\phi(x)| = v + \frac{1}{\sqrt{2}}\sigma(x)$$

The transformation of the gauge field is determined by (3.4) yielding

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$$

Inserting the above into (3.12)

$$\mathcal{L} = -\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2v^2(A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x))\right) - \left(\frac{1}{2}(\partial_\mu\sigma)^2 + 2\lambda v^2\sigma^2\right) + \mathcal{L}_{int}$$

Where $\mathcal{L}_{int}$ is an interaction term containing terms quartic and cubic in σ and $(A_\mu + \partial_\mu\alpha/e)$. Contrary to the case of global symmetry breaking the Lagrangian now contains two mass terms. We thus have a Higgs particle of mass $m_H^2 = 4\lambda\phi_0^2$ also present earlier and a vector particle of mass $m_\lambda^2 = 2e^2\phi_0^2$.

4 Solitons - Domain Walls and Vortices

The concept of a Higgs mechanism for a $U(1)$ symmetry which ended the last section has some interesting consequences.

It is normally true that our field ϕ will approach some vacuum value at infinity. If the symmetry is spontaneously broken there are several of these vacuum values, and it is therefore possible to specify some non-trivial boundary conditions at infinity. Interpolating from these boundary conditions at infinity the field must differ from the vaccum value somewhere in the interior thus giving rice to stable "lumps" of energy known as *solitons*. In the following we will describe two kinds of soliton *The Domain Wall* and the *Vortex solution*. The later chapters will then describe relations between these two solitons.

4.1 The Domain Wall

Consider the well-known $U(1)$ Higgs model of (3.9) with the same potential, and demand reflection symmetry[2], i.e. $\phi \to -\phi$. If $\mu < 0$ the symmetry is spontaneously broken or we are rather in the Higgs phase of $\phi = \phi_0$. Assume that the field is a function of one variable, x. If the theory considered is three dimensional we will just assume that the field $\phi(x, y, z)$ depends only on x. So the present in the x direction extends in the whole $y - z$ plane. Thus the name "Domain walls".

The energy density in one dimension is given by

$$\mathcal{E} \;=\; \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \lambda\phi^4 - 2\lambda\phi_0^2\phi^2 \tag{4.1}$$

[2]This symmetry is often referred to as Z_2 symmetry. In terms of Lie groups it would be a $O(1) = -1, 1$ symmetry

Normalizing so the broken state has zero energy

$$\mathcal{E}' = \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \lambda\phi^4 - 2\lambda\phi_0^2\phi^2 + \lambda\phi_0^4 = \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \lambda(\phi^2 - \phi_0^2)^2 \tag{4.2}$$

The energy is at a minimum when

$$\frac{d\phi}{dx} = 0 \quad \text{and} \quad \phi^2 = \phi_0^2$$

Yielding the vacuum values $\phi = \pm\phi_0$. If there exists a solution which interpolates between the two vacuum states we will thus encounter a spontaneous breaking of the symmetry $\phi \leftrightarrow -\phi$. So we assume that

$$\lim_{x\to\infty}\phi(x) = \phi_0, \qquad \lim_{x\to-\infty}\phi(x) = -\phi_0$$

Now using a trick due to *Bogomol'nyi* we rewrite $\mathcal{E}$ as a sum of total derivatives and squares

$$\mathcal{E}' = \frac{1}{2}\left[\left(\frac{d\phi}{dx}\right) + \sqrt{2\lambda}(\phi^2 - \phi_0^2)\right]^2 - \sqrt{2\lambda}\frac{d}{dx}\left(\frac{\phi^3}{3} - \phi_0^2\phi\right)$$

We integrate $\mathcal{E}'$ to obtain the total energy per unit area E also called the tention T. The last term is a derivative and will result in a constant contribution to the energy. The first term is a square and we can therefore find the minimal energy by minimizing this term,

$$T = E = \int_{-\infty}^{\infty} dx\mathcal{E}' = \frac{1}{2}\int_{-\infty}^{\infty} dx\left[\frac{d\phi}{dx} + \sqrt{2\lambda}(\phi^2 - \phi_0^2)\right]^2 + \frac{4}{3}\sqrt{2\lambda}\phi_0^3 \tag{4.3}$$

Minimizing the first term

$$\frac{d\phi}{dx} = -\sqrt{2\lambda}(\phi^2 - \phi_0^2) \Rightarrow$$
$$\phi(x) = \phi_0\frac{k\mathrm{e}^{2\sqrt{2\lambda}\phi_0 x} - 1}{k\mathrm{e}^{2\sqrt{2\lambda}\phi_0 x} + 1}$$

Setting $k = \mathrm{e}^{2\sqrt{2\lambda}\alpha}$ we see that k is just a parameter translating the field

$$\phi(x) = \phi_0\tanh\left(\sqrt{2\lambda}\phi_0 x\right)$$

while leaving the total energy untouched. The minimal energy consists only of the last term in (4.3)

$$E = \frac{4}{3}\sqrt{2\lambda}\phi_0^3$$

Knowing the general properties of the domain wall, we can make some qualitative predictions. For some potential $V(\phi)$ with two minima at 0 and ϕ_0, and a maximum value $max[V([0;\phi_0])] = v_0$, we have the general Lagrangian

$$L \propto -(\partial_\mu\phi)^2 + V(\phi)$$

If there is a static domain wall of width Δ interpolating between the values of 0 and ϕ_0 we must have that

$$T = E = \int dx\left[(\partial_x\phi)^2 + v(\phi)\right]$$

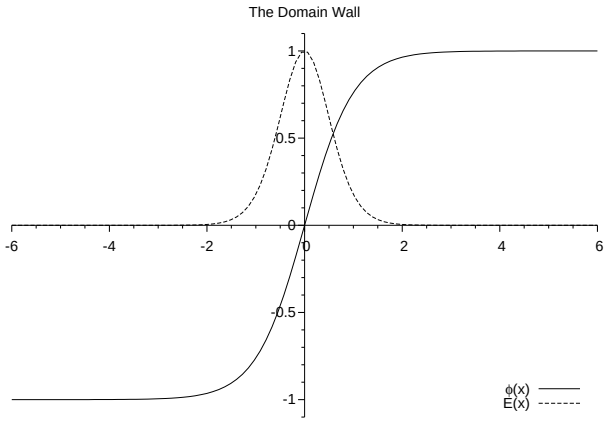

Figure 2: The domain wall in the case of the quartic potential positioned at $x = 0$. The energy density is also plotted. Note that the width of the wall seems to scale like $\Delta \propto \sqrt{\frac{2}{\lambda}}\frac{1}{\phi_0}$.

we can then make the crude assumption

$$V(\phi) \approx \begin{cases} v_0 & ,|x| < \Delta/2 \\ 0 & ,\text{elsewhere} \end{cases} \qquad (\partial_x \phi)^2 \approx \begin{cases} \left(\frac{\phi_0}{\Delta}\right)^2 & ,|x| < \Delta/2 \\ 0 & ,\text{elsewhere} \end{cases}$$

That is, approximating the domain wall as a straight line on some interval of length Δ with slope ϕ_0/Δ. We see that

$$T \approx \frac{\phi_0^2}{\Delta} + v_0 \Delta$$

When T is minimal the system is static, thus minimizing the above expression

$$T_{minimal} \approx \sqrt{v_0}\phi_0 \qquad \Delta_{minimal} \approx \frac{\phi_0}{\sqrt{v_0}} \tag{4.4}$$

A result which we will use later.

4.2 Vortices

Let us now consider a cylindrical symmetric field ϕ in a spontaneously broken symmetry. Thus the field will depend only on two coordinates x and y. In three dimensions $\phi(x, y, z)$ will not depend on z and the structure in the $x - y$ will just extend in the z direction. Therefore the name "vortex" and "fluxtube".

Using polar coordinates (r, θ) for simplicity, we assume that there exists a solution with the boundary conditions

$$\lim_{n\to\infty} \phi = \phi_0 \mathrm{e}^{in\theta} \tag{4.5}$$

where $n \in \mathbb{Z}$ is called the winding number.

Since we are forcing these boundary conditions upon the system, we cannot be sure that the solutions will be physically acceptable. One way to do this is to consider the energy per unit length in the z direction relative to the vacuum,

$$E = \int dx\, dy \left[\frac{1}{4}F_{ij}^2 + |D_i\phi(x)|^2 + \lambda(\phi^*\phi - \phi_0^2)^2\right] \tag{4.6}$$

This energy must be finite, and thus $D_i\phi(x) \to 0$ as $r \to \infty$. Using the expression (3.5) for D_i together with the boundary condition (4.5) we obtain

$$eA_i \to n\partial_i\theta(x,y) \qquad \text{for } r \to \infty \tag{4.7}$$

Meaning that A is a pure gauge. It is now clear that the boundary conditions combined with the requirement of finite energy relative to the vacuum (4.6) have fulfilled the vacuum conditions (3.13) at the limit $r \to \infty$.

Having verified the correctness of our boundary conditions, we have actually already obtained enough information to calculate the magnetic flux in the z-direction using Stokes theorem

$$\Phi_z = \int F_{12}\, d\sigma = \oint_{circle\ at\ r=\infty} A_i\, dx_i = \frac{n}{e}(\theta(2\pi) - \theta(0)) = \frac{2\pi n}{e} \tag{4.8}$$

And we vortex will in fact contain a quantified magnetic flux. The vortex solution was first discovered in by H.B. Nielsen and P. Olesen and published in [2]. It is therefore referred to as the Abrikosov-Nielsen-Olesen (ANO) vortex solution.

Inserting the Lagrangian for the vortex solution considering the general $U(1)$ Higgs model with potential $V(|\phi|)$,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\partial_\mu + ieA_\mu)\phi^*(\partial^\mu - ieA^\mu)\phi + V(|\phi|) \tag{4.9}$$

into the Euler-Lagrange equations we obtain

$$(\partial_\mu + ieA_\mu)^2\phi = \frac{1}{2}\frac{\delta V}{\delta\phi} \tag{4.10}$$

$$\partial^\nu F_{\mu\nu} = j_\mu = ie(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) + e^2A_\mu\phi^*\phi \tag{4.11}$$

From the previous discussion we make the assumption that for $n \in \mathbb{Z}$ we have the *ansatz*

$$\begin{aligned} \phi &= \mathrm{e}^{in\theta}f(r) \\ A &= \frac{n}{er}A(r)\hat{\theta} \end{aligned} \tag{4.12}$$

Where f and A are some profile functions, this ansatz will coincide with the boundary conditions. Due to [4] we can view vortex solution of several flux quanta as a superposition of unit flux vortices positioned at $x = y = 0$. The vortex solutions are therefore also known by the name *Multi-vortices*.

Inserting the above expressions into the first equation of motion we obtain

$$\begin{aligned} &(\partial_\mu + iA_\mu)^2\phi = \frac{1}{2}\frac{\delta V}{\delta\phi} && \Leftrightarrow \\ &\partial_\mu^2\phi - e^2A_\mu^2 + iA_\mu\partial_\mu\phi + i\partial_\mu A_\mu\phi_\mu] = \frac{1}{2}\frac{\delta V}{\delta\phi} && \Leftrightarrow \\ &\nabla^2\phi + e^2|A|^2\phi + 2iA\cdot\nabla\phi = \frac{1}{2}\frac{\delta V}{\delta\phi} && \Leftrightarrow \\ &\mathrm{e}^{in\theta}\left[\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{n^2}{r^2}f + \frac{n^2}{r^2}A(r)^2 - 2\frac{n^2}{r^2}A(r)\right] = \frac{1}{2}\mathrm{e}^{in\theta}\frac{\delta V}{\delta f} && \Leftrightarrow \\ &\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - n^2\frac{(1-A)^2}{r^2}f - \frac{1}{2}\frac{\delta V}{\delta f} = 0 \end{aligned} \tag{4.13}$$

Using similar tricks also rewrite the second equation of motion, obtaining

$$\frac{d^2A}{dr^2} - \frac{1}{2}\frac{dA}{dr} + 2e^2(1-A)f^2 = 0 \tag{4.14}$$

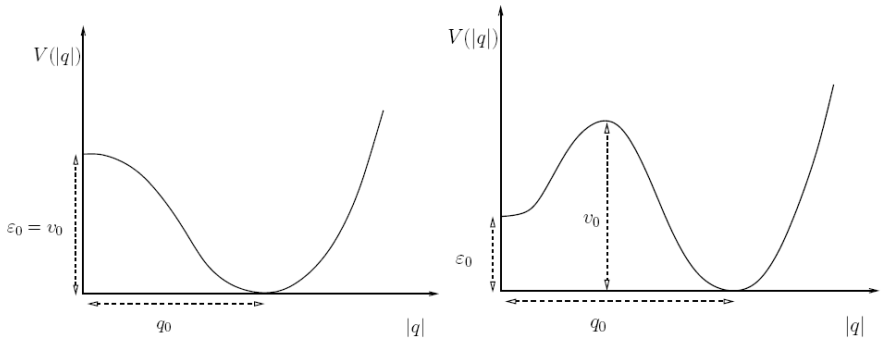

Figure 3: Either the Coulomb vacuum is stable or unstable

These equations are way to complicated to deal with analytically.

So setting $V(\phi) = \lambda(|\phi|^2 - \phi_0^2)^2$, we try applying Bogomol'nyi's trick writing the energy density $\mathcal{E}$ as a sum of squares and derivatives.

$$\begin{aligned}\mathcal{E} &= |D_1\phi|^2 + |D_2\phi|^2 + \frac{1}{2}F_{12}^2 + \lambda(|\phi|^2 - \phi_0^2)^2 \\ &= |\left(D_1 + iD_2\right)\phi|^2 - i\epsilon_{ij}\partial_i(\phi^+ D_j\phi) + \frac{1}{2}\left(F_{12} + e(|\phi|^2 - \phi_0^2)\right)^2 + e\phi_0^2 F_12 + \left(\lambda - \frac{e^2}{2}\right)(|\phi|^2 - \phi_0^2)^2\end{aligned}$$

When $\lambda = e^2/2$ we can calculate the minimum tension (under the assumption that the squares can be solved simultaneously),

$$E = \int dxdy\, \mathcal{E} = e^2\phi_0^2 \int F_{12}\, dxdy = 2n\pi\phi_0^2$$

The case where $\lambda = e^2/2$ is formally known as the *Bogomol'nyi-Prasad-Sommerfeld* case often referred to simply as the BPS case.

5 The Vortex Wall and Multi-vortices

In the previous sections I introduced the necessary theory of solitons in a $U(1)$ Higgs model, and we are now ready to discuss the concepts of [1].

The basic idea is to look at a domain wall wrapped into a cylinder and compare it to a multi-vortex solution. Our guess is simply that at a large magnetic flux the tension of the the wrapped up wall vortex and the multi-vortex will approach a common value.

We consider again the $U(1)$ Higgs model. In general the potential will interpolate between the Coulomb phase and the Higgs phase, having the value of ϵ_0 in the Coulomb phase and a maximum value of v_0 for values of ϕ in between the two phases. The situation is described in Figure 3.

Consider the domain wall interpolating between the Coulomb phase and the Higgs phase as an independent object. In this case we refer to the results of (4.4),

$$T_W \approx \sqrt{v_0}\phi_0, \qquad \Delta_W \approx \frac{\phi_0}{\sqrt{v_0}}$$

Deforming the domain wall into a cylinder of radius $R >> \Delta_W$ with the Coulomb phase on the inside and the Higgs phase on the outside we get a configuration similar to a vortex soliton. A cross section of the configuration is showed on Figure 4.

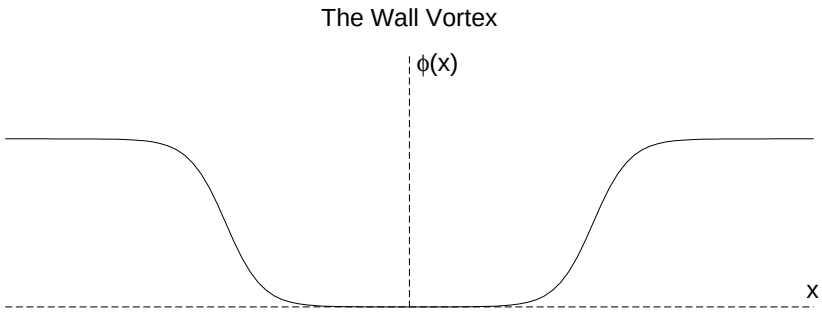

Figure 4: Pictured are a cross section in the x direction of the field configuration of a vortex wall.

It is evident that in one dimension the field can be viewed as a combination of a domain wall and an anti domain wall. Thus in on dimension there is an attractive force between the walls. To stabilize the configuration we add a quantized magnetic flux of $\Phi(n) = 2\pi n$ inside the wall tube, thus opposing the collapsing forces.
The tension due to the magnetic flux is simply

$$T_B = \int dxdy\, \mathcal{E}_B = \frac{B^2}{2e^2}\pi R^2 = \frac{2\pi n^2}{e^2 R^2}$$

Having used the relation $\Phi = B/\pi R^2$. The different contributions to the tension are thus

$\frac{2\pi n^2}{e^2 R^2}$	Tension due to the applied magnetic field
$T_W 2\pi R$	Tension of wall integrated over perimeter of vortex
$\epsilon_0 \pi R^2$	Energy density of coulomb phase integrated over interior of vortex

Assuming that in the large n limit the radius R is also large, and considering the attractive force we can neglect the second term of the tension when comparing it to the third term. This is known as the MIT limit. Thus minimizing the tension

$$T_W \approx \frac{2\pi n^2}{e^2 R^2} + \epsilon_0 \pi R^2$$

We obtain the values

$$R_{MIT} = \sqrt[4]{2}\frac{1}{\mathrm{e}^{1/2}\epsilon_0^{1/4}}\sqrt{n} \qquad T_{MIT} = 2\sqrt{2}\pi\frac{\sqrt{\epsilon_0}}{e}n \tag{5.1}$$

Checking that our assumption holds we compare the two terms of the tension responsible for attraction

$$\epsilon_0 \pi R_{MIT}^2 >> T_W 2\pi R_{MIT} \Leftrightarrow$$
$$n >> \frac{\phi_0^2 e v_0}{\epsilon_0^{3/2}}$$

Notice that $T_{MIT} \propto n$ as in the BPS case of the quartic potential. We can now quantify the behaviour that we are looking for in the following way

$$\lim_{n\to\infty} T_V(n) = T_{MIT}(n) = 2\sqrt{2}\pi\frac{\sqrt{\epsilon_0}}{e}n \tag{5.2}$$

6 The Differential Equations

Showing (5.2) we must solve the equations of motion for a vortex solution (4.13) and (4.14),

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-n^2\frac{(1-A)^2}{r^2}f-\frac{1}{2}\frac{\delta V}{\delta f}=0$$
$$\frac{d^2A}{dr^2}-\frac{1}{2}\frac{dA}{dr}+2e^2(1-A)f^2=0$$

Before proceding we would like to rescale these equations setting

$$V(\phi)=\epsilon_0\mathsf{V}\left(\frac{f}{f_0}\right)$$

where $\epsilon_0 = V(0)$ and f_0 is the value of the profile function at which the potential vanishes. This rescaled potential fulfills $\mathsf{V}(0) = 1$ and $\mathsf{V}(1) = 0$. We also scale the field so $f = f_0\chi$, reducing the equations to

$$\frac{d^2\chi}{dr^2}+\frac{1}{r}\frac{d\chi}{dr}-n^2\frac{(1-A)^2}{r^2}\chi-a\frac{\delta V}{\delta\chi}=0 \tag{6.1}$$
$$\frac{d^2A}{dr^2}-\frac{1}{2}\frac{dA}{dr}+b(1-A)\chi^2=0 \tag{6.2}$$

with the parameters

$$n,\qquad a=\frac{1}{2}\frac{\epsilon_0}{q_0^2},\qquad b=2e^2q_0^2$$

Writing R_{MIT} in terms of these

$$R_{MIT}=\sqrt[4]{2}\frac{1}{\sqrt{e}\sqrt[4]{\epsilon}}\sqrt{n}=\sqrt[4]{2}\frac{\sqrt{n}}{\sqrt[4]{ab}}$$

We observe that in the large n limit the radius will approach zero. We can avoid this inconvenient behaviour by letting the constants a and b scale with n, so

$$a_n=a_1n\qquad b_n=b_1n$$

and the radius of the wall vortex reduces to $R_{MIT}=\sqrt[4]{2/a_1b_1}$. Since $R_V >> \Delta_W$ the rescaled profile function χ must behave somewhat like a step, having $\chi\approx 0$ in the Coulomb phase $r < R_V - \Delta_W$ and $\chi\approx 1$ in the Higgs phase $r > R+\Delta_W$. In the limit $n\to\infty$ the width $\Delta_W\to 0$ and we will thus experience

$$\lim_{n\to\infty}\chi(r)=\theta_H(r-R_V)$$

We study (6.2) in the large n limit. When $r > R_V$ we have $\chi = 1$. For large r the last term in (6.2) will dominate, and the only way of getting rid of it is setting $A = 1$
When $r < R_V$ we have $\phi = 0$ and the equation reduces to

$$r\frac{d^2A}{dr^2}=\frac{dA}{dr} \tag{6.3}$$

Solving this and remembering the boundary condition $A(R_V) = 1$ we have $A=\frac{r^2}{R_V^2}$. Summarising our findings for the large n limit

$$\lim_{n\to\infty}\chi(r)=\begin{cases}0 & \text{for } r<R_V\\ 1 & \text{for } r>R_V\end{cases}\qquad \lim_{n\to\infty}A(r)=\begin{cases}\frac{r^2}{R_V^2} & \text{for } r<R_v\\ 1 & \text{for } r>R_v\end{cases}$$

Simplifying the numerical computations we have chosen $R_V = R_{MIT} = 1$ setting $a_1 b_1 = 2$. We will control the parameters through the quantity

$$\beta = 8\frac{a}{b}$$

6.1 The quartic potential

In case of the quartic potential $V(\phi) = \lambda(\phi^2 - \phi_0^2)$, we can use the fact that $\epsilon_0 = \lambda\phi_0^4$ to obtain

$$\beta = 8\frac{a}{b} = 4\frac{\lambda^2\phi_0^2}{2e^2\phi_0^2} = \frac{m_H}{m_\lambda}$$

In this special case *beta* actually measures the strength of the scalar field ϕ represented with m_H and the gauge field A of mass m_λ.

6.2 Bogomol'nyi's Trick

In the special case of the quartic potential we can also apply Bogomol'nyi's Trick, trying to justify the results of (5.1). In the BPS case the quartic potential is given by

$$V(|\phi|) = \frac{e^2}{2}(|\phi|^2 - \phi_0^2)^2$$

and the tension is proportional to the winding number n

$$T = 2\pi\phi_0^2 n$$

Inserting $\epsilon_0 = V(0) = e^2\phi_0^4/2$ into the formula for the tension of the wall vortex in the MIT limit (5.1), we see that

$$T_{MIT} = 2\sqrt{2}\pi\frac{\sqrt{\epsilon_0}}{e}n = 2\pi\phi_0^2 n$$

The fact that T_{MIT} coincides with T_{BPS} reveals a link between the wall vortex and multi-vortices. Apart from encouraging in our efforts to prove (5.2), this property of the tension will help us in our quest of solving the equations of motion numerically. Now we do not need to calculate the tension of the wall vortex. Instead we just compare the tension values of various multi-vortices with the BPS case.

6.3 The Boundary Conditions

As mentioned before the system of equations (6.1) and (6.2) cannot be solved analytically, and we will thus try a numerical approach. The numerical approach used here, actually differs from the one used in [1], where Bolognesi and Gudnason instead of rescaling the parameters a, b rescale the radius setting $\tilde{r} = r/\sqrt{n}$. That is also why some of the following mathematical results might differ from the results stated in [1].

Since we're dealing with two second order equations of two functions ϕ and A, we need four boundary conditions. These can be deduced using our knowledge of the rescaled fields χ and A. Because of our numerical approach it is convenient to put these boundary conditions in a way depending only on the fields and their first order derivative with respect to radius r.

In the limit $r \to 0$ we have $\chi \approx 0$, and the second equation (6.2) will thus reduce to the case of (6.3) revealing that $A \propto r^2$. Writing this in terms of derivatives we get the relation

$$\lim_{r\to 0} \big(2A(r) - rA'(r)\big) = 0 \tag{6.4}$$

Using this fact we turn our attention to (6.1). In the limit $r \to 0$ the slope of the potential approximates zero as discussed earlier. Since A is quadratic in r and $\chi \approx 0$ for $r \approx 0$ the term of $n^2(1-A)^2\chi/r^2$ will only contribute with $n^2\chi/r^2$ thus reducing the equation to

$$\frac{d^2\chi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{n^2}{r^2}\phi = 0$$

By a Taylor expansion we solve this equation getting the result $\chi \propto r^n$, where the divergent r^{-n} solution is discarded. This property of χ we also express using derivatives

$$\lim_{r\to 0} \big(n\chi(r) - r\chi'(r)\big) = 0 \tag{6.5}$$

Turning to the limit $r \to \infty$, we will use a trick, writing $A(r) = 1 + r\mathsf{A}(r)$ which will reduce the equations to

$$\frac{d^2\mathsf{A}}{dr^2} + \frac{1}{r}\frac{d\mathsf{A}}{dr} - \left(\frac{b\chi^2}{r^2} + 1\right)\mathsf{A} = 0$$

Assuming χ constant we recognize this as a modified Bessel's equations with the solution

$$A(r) = 1 + r\mathsf{A}(r) = 1 + crK_1(\sqrt{b}\chi r)$$

where c is a constant of integration. In the limit $r \to \infty$ the Bessel function is exponentially decaying

$$\lim_{r\to\infty} A(r) = 1 + c\frac{\sqrt{\pi r}}{2\sqrt{b}}\mathrm{e}^{-\sqrt{b}r}$$

Where we have set the value of χ to the vacuum value $\chi = 1$. We can write this behaviour[3] as

$$\lim_{r\to\infty} \left(A(r) + \frac{1}{\sqrt{b_1 n}}A'(r)\right) = 1 \tag{6.6}$$

This boundary condition is actually fulfilled if

$$A(r) \propto 1 + k\mathrm{e}^{\sqrt{b}\chi(r)r}$$

Using a method quite similar to this we obtain the last boundary condition

$$\lim_{r\to\infty} \left(\chi(r) + \frac{1}{\sqrt{8an}}\chi'(r)\right) = 1 \tag{6.7}$$

7 The Numerical Method

Having found four boundary conditions we proceed to the numerical method.

[3] Assuming that in the large r limit $\frac{1}{\sqrt{r}e^{-r}} << \sqrt{r}e^{-r}$.

7.1 Coding process

I was advised to write the actual code using `Fortran`, and knowing nothing about neither numerical analysis nor Fortran, my first idea was to solve the simple case of the ANO vortex where $n = 1$ using a rather straightforward fourth order Runge-Kutta method. This approach proved to be a disaster, since the results did not match even the simple results of [2].
I guess that the problem with fourth order Runge-Kutta is that I could not match all four boundary conditions at the same time, and so I obtained solutions oscillating around the vacuum value in the large r limit.

My next idea was to use the method of finite difference, rewriting the differential equations to some difference equations over a mesh of points, thus putting the equations on a matrix form $\mathbf{F}$. In this way the value of the fields at the mesh points could be put in a vector $\mathbf{x}$ certain values could be In this way the problem was reduced to solving a matrix equation $\mathbf{Fx} = 0$.
The problem using this method is that the differential equations (6.1) and (6.2) are non-linear. Therefore their finite difference counterparts can not be put on matrix form.

Contrary to the Runge-Kutta method, the finite difference procedure did have some appealing features. It included (as you will later see) all the boundary conditions. And having found a solution $\mathbf{x}$ for some n we could use this solution as an initial value guess if we wanted to calculate the fields at a larger n. In this way we might hope to arrive at the predicted step function in the large n limit.
So continuing the work I ended up with the method described below, which as far as I know do not have a common name.

7.2 The method

As in the case of finite difference we assume that at $r = R$ we are well outside the vortex, and both field χ and gauge field A are close to their vacuum value. Split up the interval $[0; R]$ into a mesh of N equally spaced points with separation h.
For some field configuration χ, A and some point r we define

$$A_i = A(r_i), \qquad \phi_i = \phi(r_i)$$

We define the finite difference versions of the boundary conditions (6.4), (6.5), (6.6) and (6.7)

$$P_1 = (n-1)\chi_1 - \chi_2 \tag{7.1}$$

$$P_N = \chi_N + \frac{1}{\sqrt{8na}} \frac{\chi_N - \chi_{N-1}}{h} - 1 \tag{7.2}$$

$$Q_1 = A_1 - A_2 \tag{7.3}$$

$$Q_N = A_N + \frac{1}{\sqrt{nb}} \frac{A_N - A_{N-1}}{h} - 1 \tag{7.4}$$

The finite difference versions and the differential equations (6.1) and (6.2) are defined correspondingly

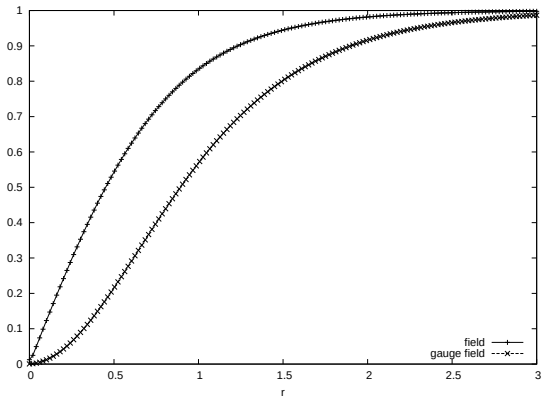

Figure 5: The numerical solution at BPS where $\beta = 1$ and $n = 1$

for $i \neq 1, N$

$$P_i = \frac{\chi_{i+1} - 2\chi_i + \chi_{i-1}}{h^2} + \frac{1}{r_i}\frac{\chi_i - \chi_{i-1}}{h} + n^2\frac{(1-A_i)^2}{r_i^2}\chi_i^2 + a\frac{\delta V}{\delta \chi}(r_i) \tag{7.5}$$

$$Q_i = \frac{A_{i+1} - 2A_i + A_{i-1}}{h^2} - \frac{1}{r_i}\frac{A_i - A_{i-1}}{h} + b(1-A_i)\chi_i^2 \tag{7.6}$$

$$\tag{7.7}$$

Let us bundle the system of finite difference equations in a vector $\mathbf{F}$ and at the same time bundle the fields in $\mathbf{x}$,

$$\mathbf{F} = (P_1 \dots P_N Q_1 \dots Q_N) \tag{7.8}$$

$$\mathbf{x} = (\chi_1 \dots \chi_N A_1 \dots A_N) \tag{7.9}$$

Generally for any functional vector we can calculate

$$\mathbf{F}(\mathbf{x} + \delta\mathbf{x}) = \mathbf{F}(\mathbf{x}) + \mathbf{J} \cdot \delta\mathbf{x} + O(\delta\mathbf{x}^2)$$

Where $\mathbf{J}$ is the Jacobian matrix. We will not calculate the Jacobian matrix but instead approximate it using finite difference with some small value δ

$$J_{ij} = \frac{\partial F_i}{\partial x_j} = \frac{F_i(\mathbf{x} + \delta \cdot \hat{\mathbf{x}_j}) - F_i(\mathbf{x})}{\delta}$$

We neglect terms of order $\delta\mathbf{x}^2$, and set $\mathbf{F}(\mathbf{x} + \delta\mathbf{x}) = 0$. Then we can find a step $\delta\mathbf{x}$ moving all of the functions of $\mathbf{F}$ close to a zero simultaneously,

$$\mathbf{J} \cdot \delta\mathbf{x} = -\mathbf{F}$$

But iterating x using this procedure will not necessarily lead us to the zero of the functions $\mathbf{F}$. Due to second order corrections we might instead just end up in some minimum. We will take further measure in order to avoid this requiring that the step decreases the function $f = |\mathbf{F}|^2 = \mathbf{F} \cdot \mathbf{F}$. It actually shows that $\delta\mathbf{x}$ is a direction of descent

$$\nabla f \cdot \delta\mathbf{x} = (\mathbf{F} \cdot \mathbf{J}) \cdot (-\mathbf{J}^{-1} \cdot \mathbf{F}) = -\mathbf{F} \cdot \mathbf{F} < 0$$

When getting close to the zero the full step $\delta\mathbf{x}$ might not minimize f. We can take care of this by not taking the full step

$$\mathbf{x}_{new} = \mathbf{x}_{old} + \lambda\delta\mathbf{x}, \qquad 0 < \lambda \leq 1$$

We are sure that for λ small enough $\mathbf{x}_{new}$ will actually decrease f, because $\delta\mathbf{x}$ is a descent direction for f. But how to choose the parameter λ. Of course it should fulfill $f(\mathbf{x}_{old}) > f(\mathbf{x}_{new})$ but this may no hinder f decreasing too slowly and in fact converging to some finite value. The solution is to require that f during an iteration at least will decrease by some fraction α of the initial decrease

$$f(\mathbf{x}_{new}) \leq f(\mathbf{x}_{new}) + \alpha\nabla f \cdot (\mathbf{x}_{new} - \mathbf{x}_{old})$$

Given some initial configuration ϕ we can thus iterate until it either is not possible to approach the minimum further of we in fact end up with a field configuration solving the finite difference equations simultaneously. More intuitively one can think of it as multiplying the vector $\mathbf{x}$ with different matrices each nudging the $\mathbf{x}$ closer to a zero.
Using two straight lines interpolating between the values of $\phi = A = 0$ and $\phi = A = 1$ as the initial guess we can solve the simple case of the vortex of unit flux i.e. $n = 1$, showing in Figure 5.
In order to reach the large n limit, we will feed the algorithm with this solution, then solving for the case $n = 2$, and continuing this way, until reaching some large n values.

7.3 The Code

The code is written in the `Fortran 90` language. It is heavily build upon the `newt` routine of [8]. It consists of some modules and subroutines

subroutine	`NEWT`	The key program. It checks whether the value of $\mathbf{x}$ returned from `LNSRCH` has converged and if not it will iterate once more.
function	`FUNCV`	Calculating the function value at a certain $\mathbf{x}$
subroutine	`LNSRCH`	The actual algorithm. Here the concept
module	`FMINLN`	Calculates the value of the function f, sharing the value with `NEWT` and `LNSRCH`
subroutine	`FDJAC`	Calculates the finite difference approximation of the jacobian
module	`TYPE`	Type definitions for different parameters (controlling the precision)
module	`INTERFACE`	A FORTRAN specific module specifying interfaces to different subroutines and functions in order to avoid type clashed, i.e. parameters have different type attributes in differed subroutines.
module	`UTIL`	A collection of some light but useful utility functions
subroutine	`LUDCMP` and `LUBKSB`	These are simply matrix subroutines solving the matrix equation of $\mathbf{J} \cdot \delta\mathbf{x} = -\mathbf{F}$ for $\delta\mathbf{x}$

7.4 The Tension - proving the conjecture

From the Lagrangian (4.9) using the ansatz (4.12) we can calculate the tension

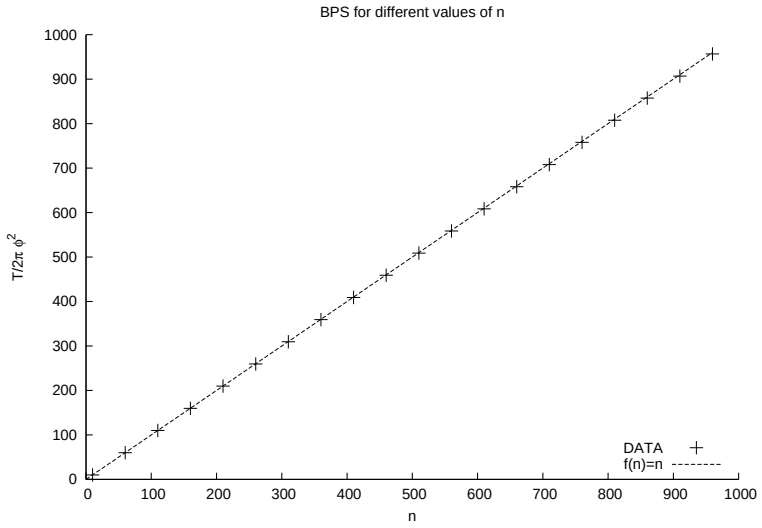

Figure 6: The tension in the BPS case

$$\begin{aligned} T_v(n) &= \int dxdy\, \mathcal{E}(n) \\ &= 2\pi \int r\, dr \left[\frac{1}{2e^2} \left(\frac{nA'}{r} \right)^2 + \frac{n^2}{r^2}(1-A)^2 q^2 + q'^2 + V(q) \right] \\ &= 2\pi\phi_0^2 \int r\, dr \left[\frac{1}{b} \left(\frac{nA'}{r} \right)^2 + \frac{n^2}{r^2}(1-A)^2 \chi^2 + \chi'^2 + 2aV(\chi) \right] \end{aligned} \tag{7.10}$$

In the follwing we will only be dealing with the quartic potential $V(\chi) = (\chi^2 - 1)^2$.

We have done the numerical analysis for three different values of β. Figure 7 show an extract of the results for the cases $\beta = 1/16$ at $n = 100$ and $n = 85.000$. Figure 8 and Figure 9 contains the same results in the cases $\beta = 1$ and $\beta = 16$ respectively.
Note that at large n the profile function χ approaches the predicted step-function independent of the value of β. At small n we see a definite difference between the profile functions for different β.
If we return to the tension, comparing the BPS case of the quartic potential, $T = 2\pi\phi_0^2 n$ to (7.10) we see that

$$T_v(n) = 2\pi\phi_0^2 \int r\, dr \left[\frac{1}{b_1 n} \left(\frac{nA'}{r} \right)^2 + \frac{n^2}{r^2}(1-A)^2 \chi^2 + \chi'^2 + 2a_1 n V(\chi) \right] = 2\pi\phi_0^2 n$$

Thus naming

$$T'_v(n) = \int r\, dr \left[\frac{1}{b_1 n} \left(\frac{nA'}{r} \right)^2 + \frac{n^2}{r^2}(1-A)^2 \chi^2 + \chi'^2 + 2a_1 n V(\chi) \right]$$

we would expect $T'_{BPS}(n) = n$ in the BPS case. This behaviour was not shown in [1], but the Figure 6 shows that our expectations are indeed fulfilled for $1 < n < 1000$. Encouraged by this result we look at the fraction

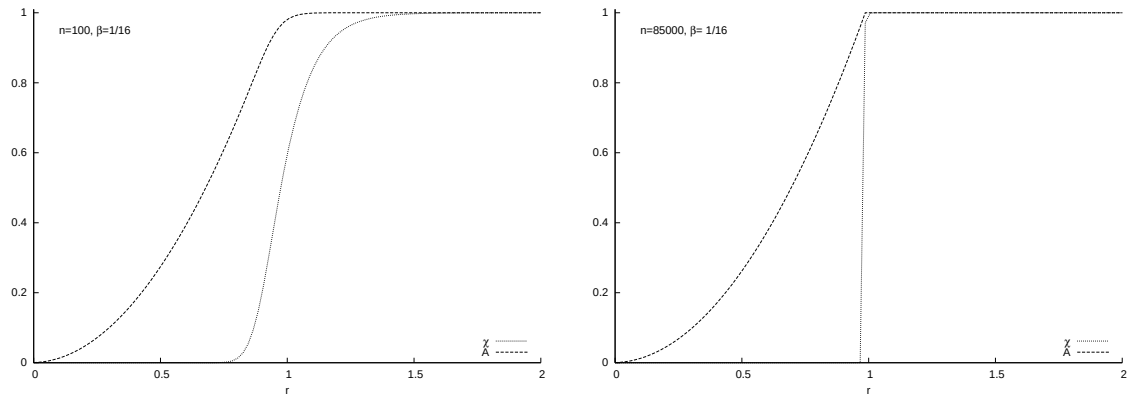

Figure 7: $\chi(r)$ and $A(r)$ plotted for $\beta = 1/16$ and n=100, n=85000

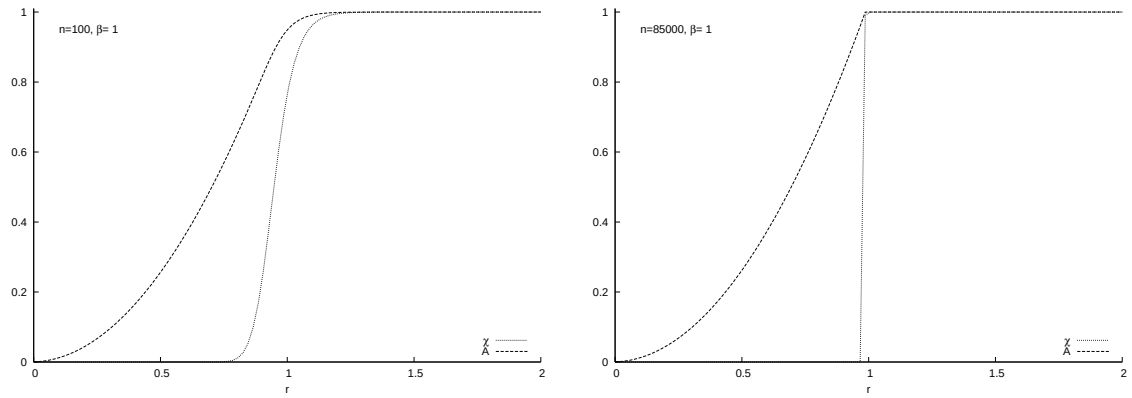

Figure 8: $\chi(r)$ and $A(r)$ plotted for $\beta = 1$ and n=100, n=85000

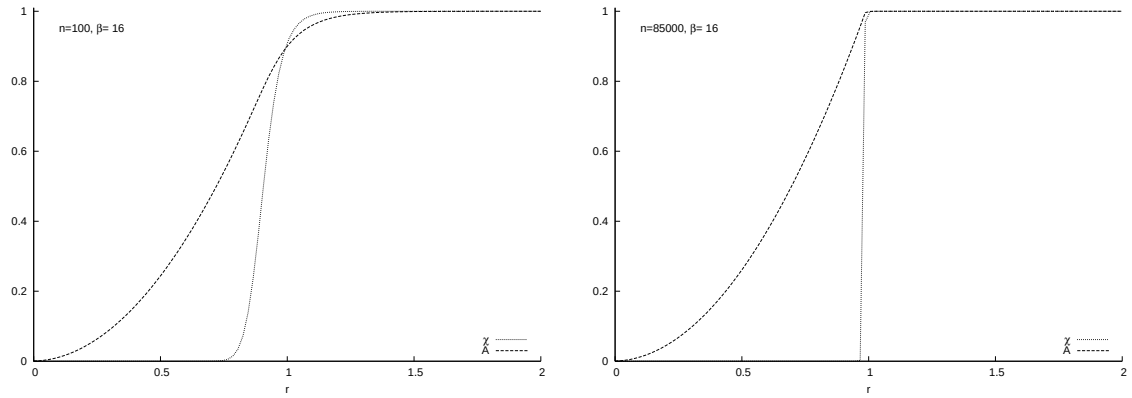

Figure 9: $\chi(r)$ and $A(r)$ plotted for $\beta = 16$ and n=100, n=25000

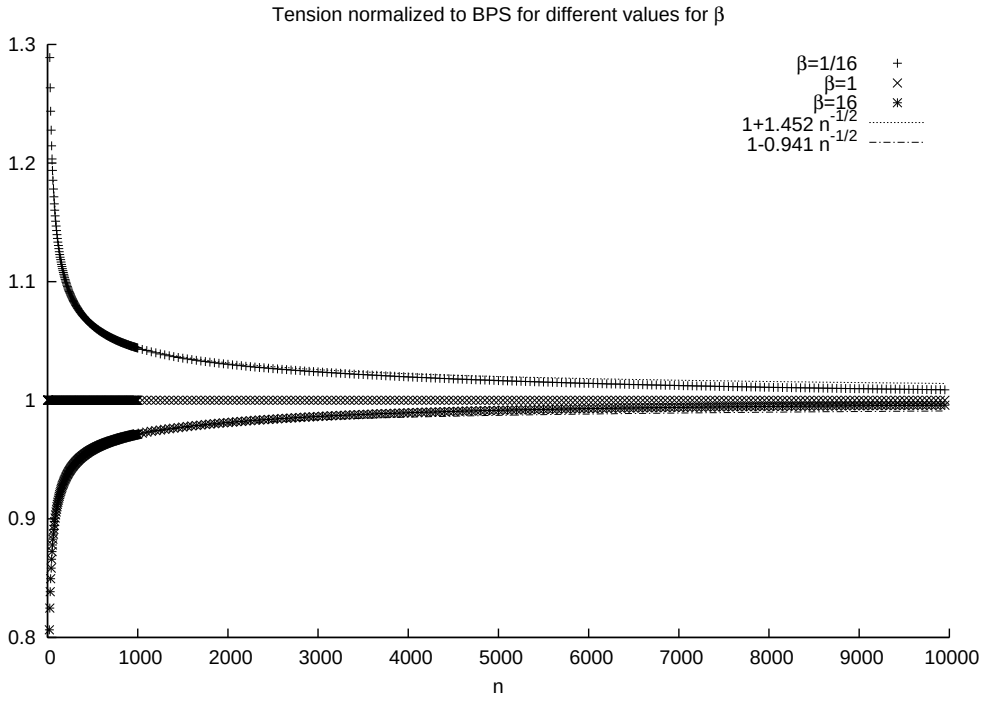

Figure 10: The tension plotted as a function of n for the cases of $\beta = 1/16, 1, 16$. The to functions shown are fitted to the results of Bolognesi and Gudnason thus comparing my results to their.

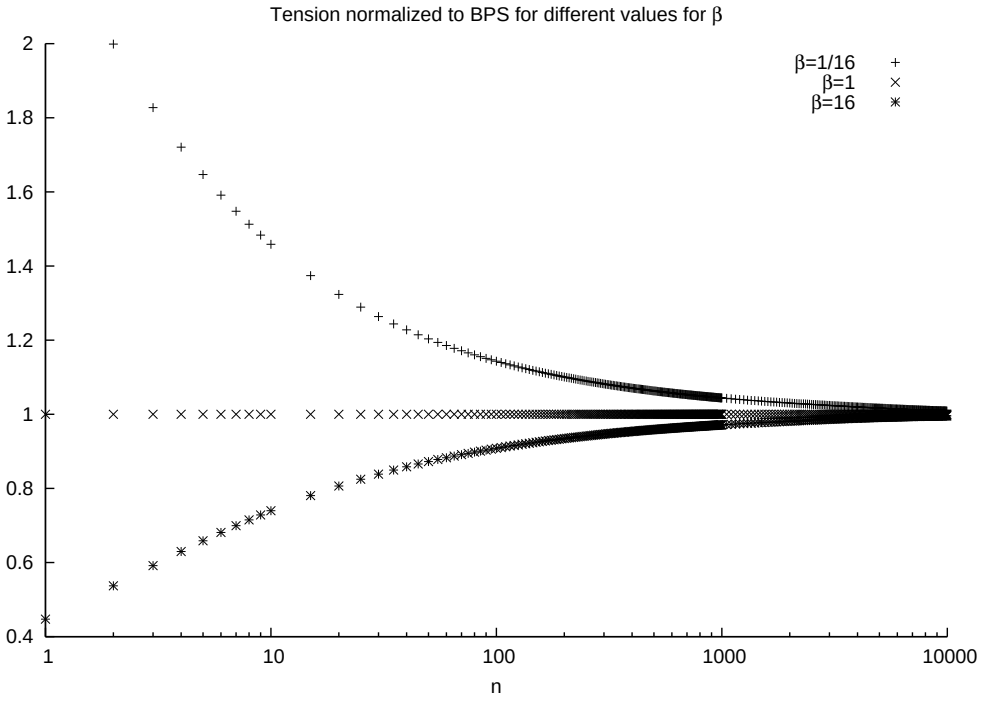

Figure 11: Tension as a function of $\log n$. Note that the values for small n does not match the values of Stefano and Gudnason.

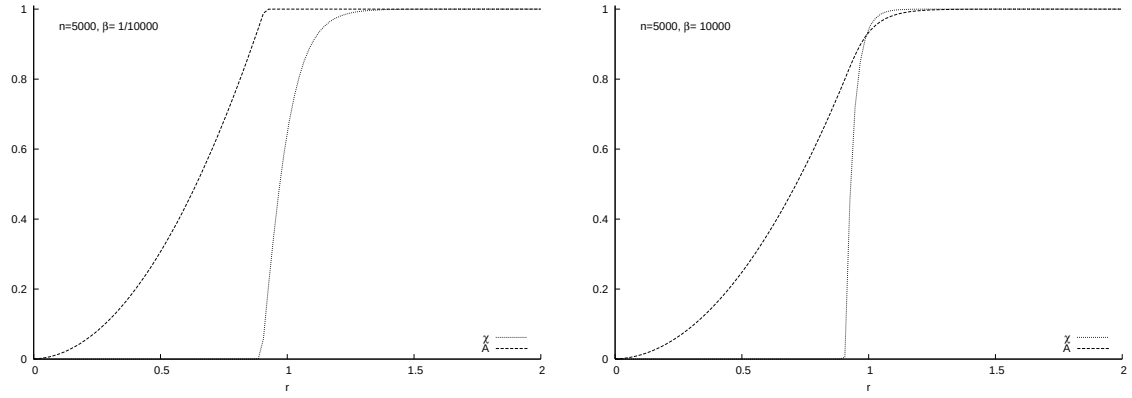

Figure 12: Showing the extreme cases og $\beta = 1/10000$ and $\beta = 10000$ at n $= 5000$

$$\mathcal{T}(n) = \frac{T_\beta(n)}{T_{BPS}(n)} = \frac{\phi_0^2 T_v'(n)}{\phi_{0,BPS}^2 T_{BPS}'} = \frac{2a_{BPS} T_v'(n)}{2a T_{BPS}'(n)}$$

Here we made the assumption that ϵ_0 is a constant not depending on the value of β. We plot $\mathcal{T}(n)$ for $\beta = 1/16, 1, 16$. The result is showed in Figure 10 and Figure 11.

Comparing my own results with the results of Bolognesi and Gudnason, it is clear that they are no perfect match. At small values my values for $\mathcal{T}$ are to far away from the BPS case while at large n they are too close.
This behaviour might be caused by two things. Either it is because Bolognesi and Gudnason rescaled the radius in order to do the numerical approximation while I scaled the parameters a and b. Or it may be caused by me not having as many mesh points as they. But since runtime is dramatically increased with the meshsize t
Not matter hat the cause might be, I do find my results close enough to their result. And it is indeed possible to read off the predicted behaviour

$$\lim_{n\to\infty} T_v(n) = \lim_{n\to\infty} T_{BPS}(n) = \lim_{n\to\infty} T_{MIT},$$

thus verifying the assumption of (5.2).

7.5 The Extreme cases

Here we will still considering the case of a ϕ^4 potential, see what happens when β is in the extreme limit of $\beta \to 0$ and $\beta \to 0$. The results are shown at Figure 12.
Note how the A field is shifted in the two cases, and the abrupt change of χ when $\chi = 0$.
We rest assured that our numerical method indeed works, since when dealing with extreme values of β still reproduces the previous results of (5.2). The tension-plot can be seen in Figure 13.

8 Linking to reality - Superconductivity

From the very beginning the ANO vortex solutions has been closely coupled to the field of superconductivity. It turns out that the Ginzburg-Landau theory describing superconductivity semi-quantum

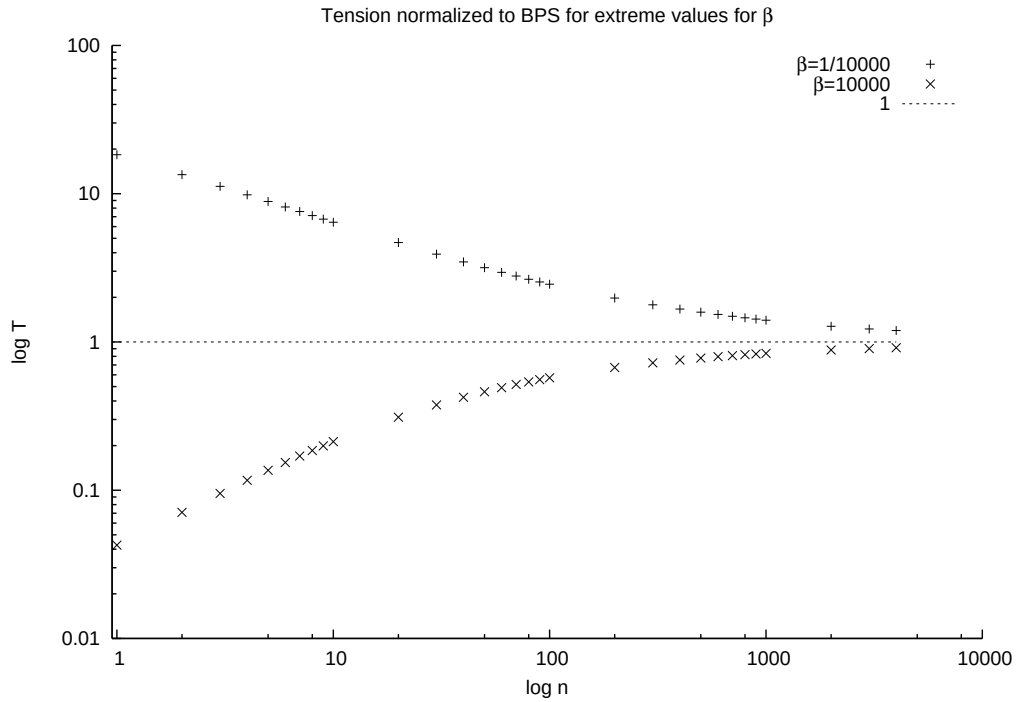

Figure 13: $logT$ as a function of $\log n$

mechanically, actually has a Lagrangian identical to (4.9), known as the Ginzburg-Landau equations. In superconductivity you have two cases. A Type I superconductor will expell any magnetic flux from the interior, while a Type II superconductor might be penetrated by tiny tube carrying magnetic flux. These tubes are of course identical to our vortex solutions. We can actually relate the theory and results to the case of superconductivity using a very qualitative approach.

Consider the force between two particles ff in both the case of a scalar field ϕ and the case of a vector field A. According to [7] the force will behave as follows

Field	ff and $\overline{f}\overline{f}$	$f\overline{f}$
scalar (Yukawa)	attractive	attractive
vector (electricity)	repulsive	attractive

Where $\overline{f}$ is the antiparticle of f. The strength of the potential scales with e^{-mr}, where m is the mass of the corresponding particle.
If we think of a unit flux vortex as a particle f, the anti particle $\overline{f}$ is then just a vortex of negative unit flux, and superpositioning a vortex and a anti-vortex will return us to the vacuum case. In this case that means that we are dealing with two different forces, and so two different masses. That means m_H corresponding to the scalar field ϕ, and m_λ corresponding to the vector field A.

So in the case of $\beta << 1$ we have $m_H << m_\lambda$ meaning that the force arising from the gauge field A will dominate the force from the scalar field ϕ. Thus there will be repulsive force between some unit flux vortices. In superconductivity this is similar to the case of Type II superconductors, because the

repelling of vortices will result in the formation of a lattice consisting of unit flux vortices.
Because of this repulsion there it will be quite impossible to observe a multi-vortex.

In the case of $\beta >> 1$ we will instead have $m_H >> m_\lambda$, and the force due to the scalar field ϕ will dominate. In this case we have the case of Type I superconductors, because there is a attraction between the unit flux vortices. If this seems odd, then you might consider that the attraction will hinder flux-pinning, and in a superconductor vortices will thus be expelled from the interior.

Anyway it is difficult to generalize our results to the case of a lump of superconducting material since we are dealing with a system of infinite size. But it seems, that should we have any chance observing an actual multi-vortex it would be in the case of Type I superconductors, since the unit flux vortices will attract each other. So to observe a mutivortex we would need to go to the extreme London limit $\beta \approx 0$, as described with in the previous section of $\beta = 1/10000$.

9 Conclusion

We have introduced the theory of fields along with the concepts of spontaneous symmetry breaking, leading to the formation of soliton. For a wall vortex we have shown that

$$R_{MIT} = \sqrt[4]{2}\frac{1}{\mathrm{e}^{1/2}\epsilon_0^{1/4}}\sqrt{n} \qquad T_{MIT} = 2\sqrt{2}\pi\frac{\sqrt{\epsilon_0}}{e}n$$

And we have conjectured that at large magnetic flux (large n) a wall vortex and a vortex solution will be very similar. In the case of the quartic potential this is actually similar to

$$\lim_{n\to\infty} T_v(n) = \lim_{n\to\infty} T_{BPS}(n) = \lim_{n\to\infty} T_{MIT}$$

Then solving the rewritten equations of motion

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - n^2\frac{(1-A)^2}{r^2}f - \frac{1}{2}\frac{\delta V}{\delta f} = 0$$
$$\frac{d^2A}{dr^2} - \frac{1}{2}\frac{dA}{dr} + 2e^2(1-A)f^2 = 0$$

using a numerical method best described as the non-linear counterpart to finite differencing, we have found the field configurations at large n in the case of the ϕ^4 potential. And calculating the tension we did in fact show that the tension did approach the BPS value at large n, allthough the results didn't match Bolognesi and Gudnason's exact.
Then we have looked at the case of extreme β still exposing the same behaviour. And last we have made a real-life perspective to the theory developed in this project, relating it to superconductivity.

References

[1] S. Bolognesi and S. B. Gudnason, "Multi-vortices are wall vortices: A numerical proof", Nucl. Phys. B **741** (2006) 1 [arXiv:hep-th/0512132].

[2] H. B. Nielsen and P. Olesen, "Local Field Theory Of The Dual String", Nucl. Phys. B **57** (1973) 367.

[3] H. J. de Vega and F. A. Schaposnik, "A Classical Vortex Solution Of The Abelian Higgs Model", Phys. Rev. D **14**, 1100 (1976).

[4] E. J. Weinberg, "Multivortex Solutions Of The Ginzburg-Landau Equations", Phys. Rev. D **19**, 3008 (1979).

[5] P. Olesen and J. Ambjoern, "General relativity cosmology and classical gauge theories", Lecture notes.

[6] S. Coleman, "Aspects of symmetry", Cambridge University Press 1985.

[7] M. E. Peskin and D. V. Schroeder "An introduction to Quantum Field Theory", Westview Press 1995.

[8] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery "Numerical Recipes in Fortran 77 - Second Edition", Cambridge University Press 1992.

[9] T. Vachaspati, "Kinks and Domain Walls - An introduction to Classical and Quantum Solitions", Cambridge University Press 2006.

[10] D. J. Griffiths "Introduction to Electrodynamics", Prentice-Hall International 1999.